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Stability and Optimal Feedback Controls for Time-Delayed Linear Periodic Systems

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An effective numerical method for stability analysis of feedback controls with time delay and for identifying optimal gains of the control is presented. The method develops a mapping of the system response in a finite dimensional state space. Minimization of the largest absolute value of eigenvalues of the mapping leads to optimal control gains. Numerical examples of both time-invariant and periodic linear systems are presented to demonstrate the method. We have found that the proposed method provides accurate stability boundaries and performance contours in the parametric space of control gains.

I. Introduction

TIME delay is a common nonlinear characteristic of engineering systems and is frequently a source of instability of control systems. Effects of time delay on the stability and performance of control systems have been a subject of many studies. Many aerospace applications also involve time delays. Yang and Wu have studied structural systems with time delay.[1] Space teleoperation is notorious for time delays.[2] In Ref. 3, a time-delay filter is developed to design a fuel/time optimal control. A sampled-data control system is studied in Ref. 4 with a consideration of computational time delay. The time-delay feedback control is designed in Ref. 5 to regulate the librational motion of gravity-gradient satellites in an elliptic orbit. For time-invariant linear systems with time delay, several methods are available for the design of proportional-integral-derivative (PID) controls and their stability analysis. These methods including root locus and Nyquist criterion are quite mature. A survey of more advanced stability analysis for delayed linear systems is presented in Ref. 6. The Smith predictor is a well-known method[7] that proposes a compensator to stabilize the feedback control designed for the system without time delay. Olgac and his associates have published extensively on the use of delayed resonator for vibration suppression (for example, see Ref. 8). A study on stability and performance of feedback controls with multiple time delays is reported in Ref. 9 by considering the roots of the closed-loop characteristic equation. Over the years, researchers have come to a realization that the model predictive control offers a good tool to deal with time delay.[10–12] Stability conditions of delayed time-varying systems have been extensively studied in the literature. The Lyapunov approach is a popular method to use.[13,14] A non-Lyapunov based stability study of linear time-varying system by the Gauss–Seidel iteration is presented in Ref. 15. An unconditional stability criterion is derived in Ref. 16 for time-varying discrete systems. We have found little studies on delineation of the stability domain in the parametric space of control gains for periodic linear systems with time delay. This paper presents a semidiscretization numerical method for identifying the stability region of periodic linear systems with time delay and for finding the optimal feedback gains of the system.

Semidiscretization is a common technique in structural dynamics and fluid mechanics. The method proposes to discretize some spatial or temporal variables and to treat the rest of them as continuous variables. Insperger and Stepan[17] extended this idea to the delayed ordinary differential equations. For a linear system of one degree of freedom with time delay, the system lies in an infinite dimensional state space because of the time delay. The method proposed by Insperger and Stepan discretizes the time variable and approximates the state space with one of a finite dimension. In this paper, we apply this method to the stability and performance analysis of feedback controls of periodic linear systems with time delay.

The paper is organized as follows. Section II introduces the semidiscretization method and its application to the design problem of feedback controls. In Sec. III, two examples are presented to demonstrate the application of the method.

II. Method

Consider a second-order periodic system with time delay under a proportional-derivative (PD) control given by

$$\ddot{x}(t) + a_1(t)\dot{x}(t) + a_2(t)x(t) = -k_p x(t-\tau) - k_d \dot{x}(t-\tau) \quad (1)$$

where $x(t)$ is the system response, the coefficients $a_1(t)$ and $a_2(t)$ are periodic functions of time with period T, τ is a constant time delay, and k_p and k_d are the proportional and derivative gains. In the rest of the paper, we shall consider the PD control only, although the method is applicable to PID controls.

Because of the time delay, the state vector of this simple system is no longer just $(x(t), \dot{x}(t))$, but $(x(t), \dot{x}(t), x(t-\tau_1), \dot{x}(t-\tau_1))$ for all $0 < \tau_1 \le \tau$, which has an infinite dimension. The time delay significantly complicates the solution process of the system.

The semidiscretization method is introduced next. Let us discretize the period T into an integer k intervals of length Δt such that $T = k\Delta t$. For the sake of simplicity, we assume that the time delay $\tau = n\Delta t$, where n is an integer. When an integer n cannot be found, discretization of the time delay τ will be approximate. Details on how to treat this case can be found in Ref. 17.

Consider Eq. (1) in a time interval $t \in [t_i, t_{i+1}]$, where $t_i = i\Delta t$, $i = 0, 1, 2, \ldots, k$. In each small time interval $[t_i, t_{i+1}]$, the delayed responses $x(t-\tau)$ and $\dot{x}(t-\tau)$ on the right-hand side of the equation and the time-dependent coefficients on the left-hand side of the equation are assumed to be constant. We denote

$$x(t_i - \tau) = x((i-n)\Delta t) = x_{i-n}$$

$$\dot{x}(t_i - \tau) = \dot{x}((i-n)\Delta t) = \dot{x}_{i-n}$$

$$a_1(t_i) = a_{1i}, \qquad a_2(t_i) = a_{2i} \quad (2)$$

Equation (1) becomes time invariant over this time interval:

$$\ddot{x}(t) + a_{1i}\dot{x}(t) + a_{2i}x(t) = -k_d\dot{x}_{i-n} - k_p x_{i-n} \equiv f_{i-n} \qquad t \in [t_i, t_{i+1}] \quad (3)$$

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The solution of this system can be obtained readily. When $a_{2i} \neq 0$ and $a_{1i}^2 \neq 4a_{2i}$, the general solution of the equation is

$$x(t) = c_1 e^{r_1(t-t_i)} + c_2 e^{r_2(t-t_i)} + f_{i-n}/a_{2i}, \qquad t \in [t_i, t_{i+1}] \quad (4)$$

where r_1 and r_2 are distinct roots of the characteristic equation. Coefficients c_1 and c_2 are determined by initial conditions $x(t_i) = x_i$ and $\dot{x}(t_i) = \dot{x}_i$. The response variables $x_{i+1} = x(t_{i+1})$ and $\dot{x}_{i+1} = \dot{x}(t_{i+1})$ at time t_{i+1} are then expressed in terms of the initial conditions and f_{i-n} in the following mapping:

$$\begin{Bmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{Bmatrix} = \begin{bmatrix} \alpha_{1i} & \alpha_{2i} \\ \beta_{1i} & \beta_{2i} \end{bmatrix} \begin{Bmatrix} x_i \\ \dot{x}_i \end{Bmatrix} + \begin{Bmatrix} \alpha_{3i} \\ \beta_{3i} \end{Bmatrix} f_{i-n} \quad (5)$$

where

$$\alpha_{1i} = \frac{r_1 e^{r_2 \Delta t} - r_2 e^{r_1 \Delta t}}{r_1 - r_2}, \qquad \alpha_{2i} = \frac{e^{r_1 \Delta t} - e^{r_2 \Delta t}}{r_1 - r_2}$$

$$\beta_{1i} = \frac{r_1 r_2}{r_1 - r_2}(e^{r_2 \Delta t} - e^{r_1 \Delta t}), \qquad \beta_{2i} = \frac{r_1 e^{r_1 \Delta t} - r_2 e^{r_2 \Delta t}}{r_1 - r_2}$$

$$\alpha_{3i} = \frac{1}{a_{2i}}(1 - \alpha_{1i}), \qquad \beta_{3i} = -\frac{1}{a_{2i}}\beta_{1i} \quad (6)$$

The solutions for special cases when $a_{2i} = 0$ or $a_{1i}^2 = 4a_{2i}$ or $r_1 = r_2$, etc., are well known and are not presented here for the sake of space. The mapping for all of these cases will have the same form as in Eq. (5) with different coefficients α and β.

When $n = 0$, $f_i = -k_d \dot{x}_i - k_p x_i$. Within the framework of semidiscretization, we can define an $(n+2)$ dimensional state vector as

$$\boldsymbol{y}_i = \{\dot{x}_i \quad x_i \quad f_{i-1} \quad \cdots \quad f_{i-n}\}^T \quad (7)$$

A mapping of the state vector over the interval $[t_i, t_{i+1}]$ can be found as

$$\boldsymbol{y}_{i+1} = \boldsymbol{A}_i \boldsymbol{y}_i \quad (8)$$

where the transition matrix from time t_i to t_{i+1} is

$$\boldsymbol{A}_i = \begin{bmatrix} \beta_{2i} & \beta_{1i} & 0 & 0 & \cdots & 0 & \beta_{3i} \\ \alpha_{2i} & \alpha_{1i} & 0 & 0 & \cdots & 0 & \alpha_{3i} \\ -k_d & -k_p & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad (9)$$

The mapping of the state vector over one period $T = k\Delta t$ is therefore

$$\boldsymbol{y}_{j+1} = \boldsymbol{\Phi} \boldsymbol{y}_j \quad (10)$$

where the mapping matrix $\boldsymbol{\Phi}$ is given by

$$\boldsymbol{\Phi} = \boldsymbol{A}_{k-1} \boldsymbol{A}_{k-2} \cdots \boldsymbol{A}_1 \boldsymbol{A}_0 \quad (11)$$

The index j ($j = 0, 1, \cdots$) refers to the number of periods, that is, $\boldsymbol{y}_j$ is the state vector at the beginning of the jth period.

The stability of the control system is determined by the eigenvalues of $\boldsymbol{\Phi}$. Let $\lambda_{\max}$ denote the maximum absolute value of eigenvalues of the matrix $\boldsymbol{\Phi}$. Then,

$$|\boldsymbol{y}_{j+1}| \leq \lambda_{\max} |\boldsymbol{y}_j|, \qquad \lambda_{\max} = \max(|\lambda(\boldsymbol{\Phi})|) \quad (12)$$

When $\lambda_{\max} < 1$, $\boldsymbol{\Phi}$ is a contraction, and the control system is asymptotically stable. The stability boundary is given by $\lambda_{\max} = 1$. Equation (12) indicates that the smaller $\lambda_{\max}$ is, the faster the system converges to zero. $\lambda_{\max}$ therefore also provides a measure of the control performance.

If we restrict our interest in a finite region in the parametric space (k_p, k_d) where the system is stable, we can find an optimal pair of control gains (k_p, k_d) in the region to minimize $\lambda_{\max}$. This leads to the following optimization problem for optimal gains:

$$\min_{k_p, k_d}[\lambda_{\max}(k_p, k_d)] = \min_{k_p, k_d}[\max_{n+2}(|\lambda(\boldsymbol{\Phi})|)] \qquad \text{subject to } \lambda_{\max} < 1 \quad (13)$$

This optimization formulation offers a different approach to the design of feedback controls for linear systems with time delay. The control performance criterion is the decay rate of the mapping $\boldsymbol{\Phi}$ over one period. This criterion emphasizes the overall damping and stability of the system and might not guarantee the transient performance of the system as in the case of classical PID control design.

Constraints can be imposed to the minimization problem such as the bounds of the control authority. In this preliminary work, however, we shall not study the ramifications of different constraints.

There exist a large number of solution techniques for optimization problems such as the present one. In this work, we shall apply the well-known Lagrange–Newton algorithm to search for the optimal gains.[18] The stability and convergence of the Lagrange–Newton method have been extensively studied in the literature. Because the method is well known, we shall not describe it here.

III. Examples

A. Time-Invariant System

To validate the method, we first apply it to a second-order autonomous system with time delay:

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = -k_d \dot{x}(t-\tau) - k_p x(t-\tau) \quad (14)$$

where ζ is the damping ratio and ω_n is the natural frequency. For a given gain k_p, for example, we can find a range of k_d that leads to a stable closed-loop system by using the Nyquist criterion. Because the system is autonomous, we can arbitrarily select a period $T > \tau$ to construct the mapping. For convenience, we choose the undamped natural period of the system as T. The matrices $\boldsymbol{A}_i$ in Eq. (9) are the same for all of the time intervals in a period. The stability of the system can also be determined by the eigenvalues of a single matrix $\boldsymbol{A}_i$. The simulation parameters are chosen as $\zeta = 0.05$, $\omega_n = 2$, and $n = 50$. The undamped period of the system is $T = \pi$.

Figure 1 shows the upper and lower bounds of the control gain k_p as a function of the time delay τ when $k_d = 0$. This result is fully agreeable to that obtained by the Nyquist criterion. Note a periodic change of the bounds related to T in the figure. The bounds become narrower as the time delay increases. In Ref. 8, a similar result is achieved. This trivial example helps to study the issues such as the accuracy of semidiscretization as a function of the time step Δt.

In Fig. 2, the stability boundaries on the $k_p - k_d$ plane are plotted for the system with different time delays. The size of the stability region decreases when the time delay increases. Figure 3 shows the performance contours of the feedback control as measured by the maximum absolute value of eigenvalues of the mapping matrix $\boldsymbol{\Phi}$ for a time delay $\tau = \pi/2$. This chart clearly indicates that for a given delay there is a finite optimal pair of control gains k_p and

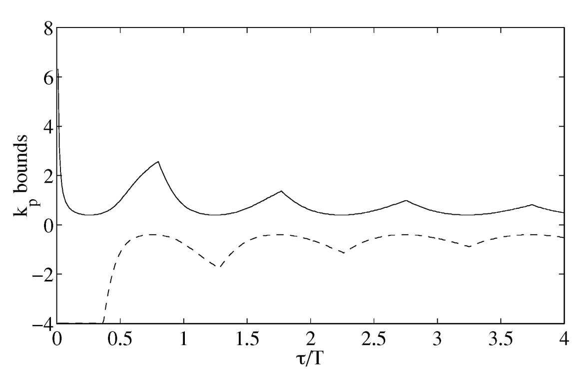

Fig. 1 Upper and lower bounds of control gain k_p vs time delays when $k_d = 0$: ——, upper bound and – – –, lower bound.

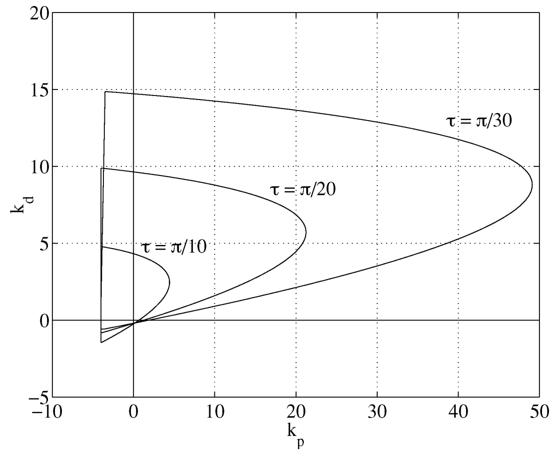

Fig. 2 Stability boundaries on the k_p–k_d plane of the time-invariant system with different time delays. The inner part of the closed contour is the stable region.

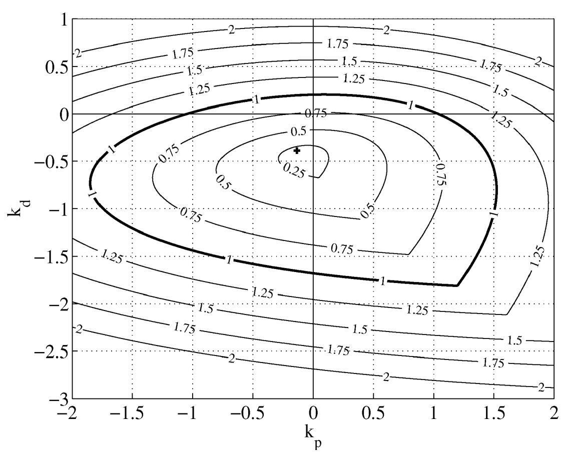

Fig. 3 Performance contours of the second-order linear time-invariant system with a time delay $\tau = \pi/2$. The labels of the contours are the maximum absolute value $\lambda_{\max}$ of eingenvalues of the mapping Φ: +, optimal control gains $(k_p, k_d) = (-0.1356, -0.3898)$ with the smallest $\lambda_{\max} = 0.1103$.

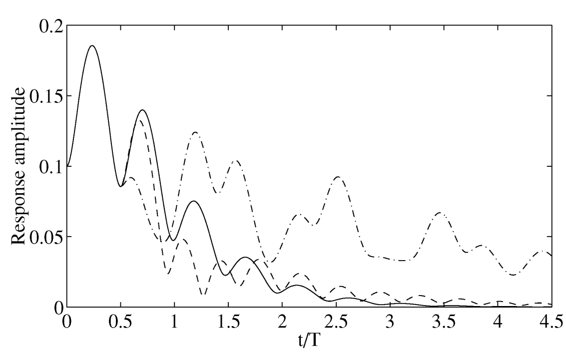

Fig. 4 Time history of the response amplitude of the autonomus system: ——, optimal gains $(k_p, k_d) = (-0.1356, -0.3898)$ and $\lambda_{\max} = 0.1103$; – – –, $(k_p, k_d) = (-0.5, -0.8)$ and $\lambda_{\max} \approx 0.5$; –·–·, $(k_p, k_d) = (0.8, -1.45)$ and $\lambda_{\max} \approx 0.75$.

k_d that will lead to the highest decay rate with the smallest $\lambda_{\max}$. Solving for the optimization problem in Eq. (13), we have found the optimal gains to be $(k_p, k_d) = (-0.1356, -0.3898)$ with $\lambda_{\max} = 0.1103$.

Figure 4 shows a comparison of time histories of the amplitude of the response vector $(x, \dot{x})$ of the system under the optimal and nonoptimal feedback controls. The optimal control provides better performance than that of the nonoptimal ones in terms of the steady-state response and its decay rate.

B. Periodic System

The strength of the proposed method lies in its ability to handle periodic systems. To demonstrate this capability, we consider the classical Mathieu equation with a delayed feedback control:

$$\ddot{x}(t) + (\delta + 2\varepsilon \cos 2t)x(t) = -k_d\dot{x}(t - \tau) - k_p x(t - \tau) \quad (15)$$

Referring to Eq. (1), we have $a_1(t) = 0$ and $a_2(t) = \delta + 2\varepsilon \cos 2t$. The period of the system is $T = \pi$. We select the parameters to be $\varepsilon = 1$, $\delta = 1$, 3, and 4. When $\delta = 1$ and 4, the uncontrolled system is parametrically unstable. Consider a delay $\tau = \pi/4$, and choose $n = 20$.

Figures 5 and 6 present the stability boundaries on the $k_p - k_d$ plane with different parameters. As is in the case of the time-invariant system, the size of the stability region decreases with the increase of time delay. The shape of the stability region is more complex than that of the time-invariant system. The irregular geometry of the contours in the figures is a reflection of the complex behavior of the time-varying system.

In the following discussions, the parameters are fixed to be $\delta = 4$, $\varepsilon = 1$, and $n = 20$. Figure 7 shows the performance contours of the control system with a time delay $\tau = \pi/4$. For the periodic system, there is also a finite optimal pair of the feedback gains that will lead to the best control performance as measured by $\lambda_{\max}$. We have found the optimal gains to be $(k_p, k_d) = (-2.0169, -0.3091)$ with $\lambda_{\max} = 1.5535e - 3$.

Figure 8 shows a comparison of time histories of the amplitude of the response vector $(x, \dot{x})$ of the system under the optimal and nonoptimal feedback controls.

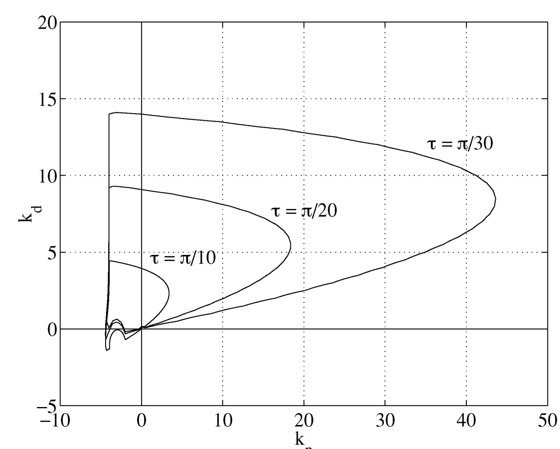

Fig. 5 Stability boundaries on the k_p–k_d plane for the periodic system with different time delays. The inner part of the closed contour is the stable region: $\delta = 4$ and $\varepsilon = 1$.

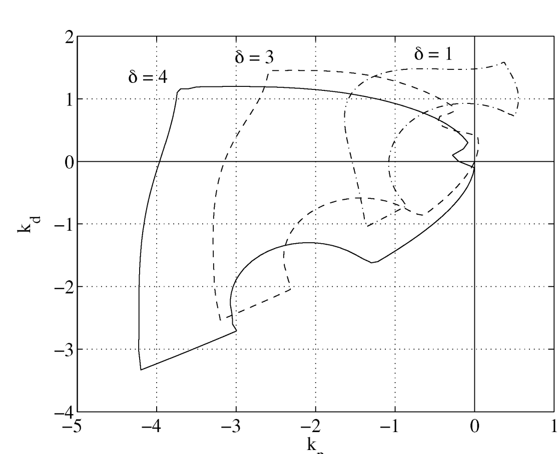

Fig. 6 Stability boundaries of the periodic system with a time delay $\tau = \pi/4$: ——, $\delta = 4$ and $\varepsilon = 1$; – – –, $\delta = 3$ and $\varepsilon = 1$; –·–·, $\delta = 1$ and $\varepsilon = 1$. The inner part of the closed contour is the stable region.

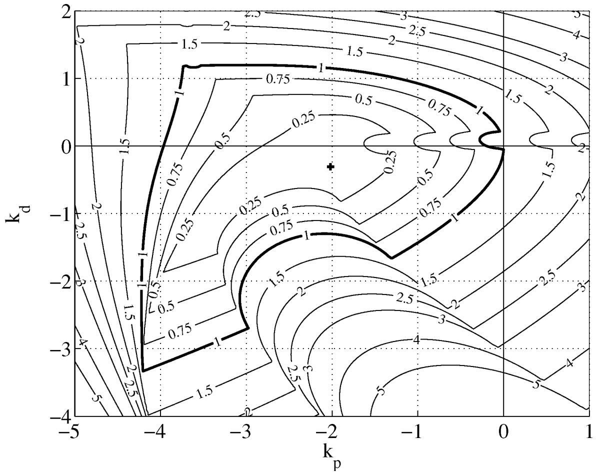

Fig. 7 Performance contours of the closed-loop periodic system with a time delay $\tau = \pi/4$. The labels of the contours are the maximum absolute value λ_{max} of eingenvalues of the mapping Φ: +, optimal control gains $(k_p, k_d) = (-2.0169, -0.3091)$ with the smallest $\lambda_{max} = 1.5535e-3$.

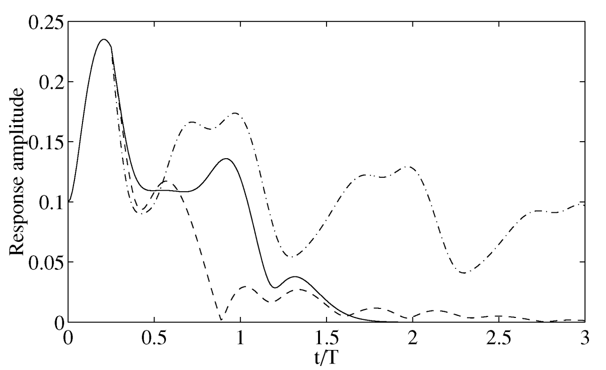

Fig. 8 Time history of the response amplitude of the periodic system: ——, optimal gains $(k_p, k_d) = (-2.0169, -0.3091)$ and $\lambda_{max} = 1.5535e-3$; – – –, $(k_p, k_d) = (-2.0, 0.4)$ and $\lambda_{max} \approx 0.25$; –·–·, $(k_p, k_d) = (-4.0, -1.0)$ and $\lambda_{max} \approx 0.75$.

C. Discussion

Both of the examples suggest that when the time delay is large enough, feedback controls with negative gains can stabilize the system and can even provide very good performance as demonstrated in the paper.

In principle, the present method can be extended to general time-varying linear systems. In this case, the mapping of the state vector over one mapping interval becomes

$$\boldsymbol{y}_{j+1} = \boldsymbol{\Phi}(j)\boldsymbol{y}_j \tag{16}$$

where the matrix $\boldsymbol{\Phi}(j)$ is now a function of the mapping step j. The asymptotic stability of the system requires

$$\lambda_{max}\left[\prod_{j=0}^{\infty}\boldsymbol{\Phi}(j)\right] < 1 \tag{17}$$

The computation to delineate the stability boundary is far more intensive. A stringent sufficient condition for asymptotic stability is that there exists a $J < \infty$ such that

$$\lambda_{max}[\boldsymbol{\Phi}(j)] < 1 \qquad \text{for all} \qquad j \geq J \tag{18}$$

The extension of the method to nonlinear systems is nontrivial and will lead to a nonlinear mapping $\boldsymbol{y}_{j+1} = \boldsymbol{F}(j, \boldsymbol{y}_j)$ with a high dimension. It would be very difficult just to locate all of the equilibrium points of the mapping in the high-dimensional state space.

The accuracy of the present semidiscretization method is examined by using the time-invariant example with known stability results. A theoretical analysis of convergence and stability of the method for general periodic systems is not available at this time and should be pursued in the future.

IV. Conclusions

The semidiscretization method is efficient and accurate in generating stability boundaries for linear time-invariant and periodic systems under a feedback control with time delay. The performance contours presented in this paper provide the stable control gains and the associated performance as measured by the largest absolute value λ_{max} of eigenvalues of the mapping $\boldsymbol{\Phi}$. The present method also provides a proper platform for identifying optimal control gains that minimize λ_{max}. The nature of conditional stability of delayed linear systems seems to guarantee the existence and uniqueness of such optimal feedback gains. A theoretical confirmation of this assertion deserves further attention. The present study has only considered proportional and derivative controls. However, it is straightforward to include the integral control. In this preliminary study, we have only considered one control design criterion to minimize λ_{max}. Implications of this design in terms of the closed-loop poles and the control design for better tracking and transient performance within the framework of semidiscretization deserve further studies.

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